PY4T01 Condensed Matter Theory: Lecture 15

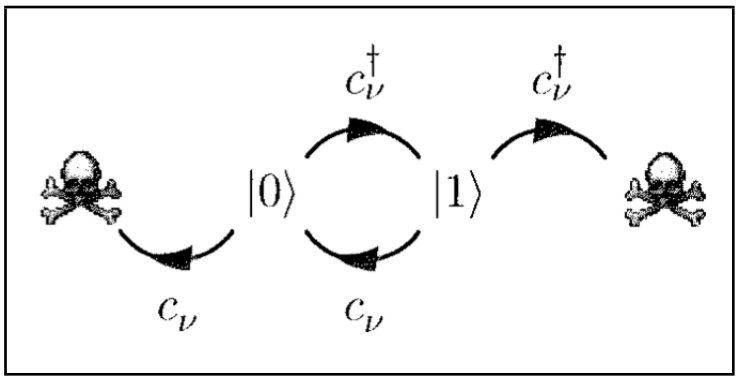
Introducing electron-electron interaction

Let us go back to the H_2 molecule and re-write H as (second quantization formalism)

$$H = \epsilon_0 \sum_{\nu=1}^{2} c_\nu^\dagger c_\nu + \gamma \left[c_1^\dagger c_2 + c_2^\dagger c_1 \right]$$

The operators c_ν and $c_\mu^\dagger$ are defined as

$$\begin{aligned} c_\nu|0_\nu\rangle &= 0 & c_\nu^\dagger|0_\nu\rangle &= |1_\nu\rangle \\ c_\nu|1_\nu\rangle &= |0_\nu\rangle & c_\nu^\dagger|1_\nu\rangle &= 0 \end{aligned}$$



A particular electronic configuration of the H_2 molecule is written in terms of the vectors

$$|\alpha_1, \alpha_2\rangle \quad \text{where} \quad \alpha_\nu = 0, 1$$

H acts as following:

1. The first term "counts" electrons

$$c_1^\dagger c_1 |\alpha_1, \alpha_2\rangle = \hat{n}_1 |\alpha_1, \alpha_2\rangle = \alpha_1 |\alpha_1, \alpha_2\rangle$$

2. The second term "transfers" electrons

$$c_1^\dagger c_2 |0_1, 1_2\rangle = |1_1, 0_2\rangle$$

The problem can be then solved as usual.

1. Take a generic wave-function written on this "new" basis

$$|\psi\rangle = \psi_1 |1_1, 0_2\rangle + \psi_2 |0_1, 1_2\rangle$$

2. Expand the Schrödinger equation

$$H|\psi\rangle = \epsilon_0 \sum_{\nu=1}^{2} c_\nu^\dagger c_\nu |\psi\rangle + \gamma \left[c_1^\dagger c_2 + c_2^\dagger c_1 \right] |\psi\rangle = E|\psi\rangle$$

$$\epsilon_0 \sum_{\nu=1}^{2} c_\nu^\dagger c_\nu |\psi\rangle = \epsilon_0 |\psi\rangle$$

$$\gamma \left[c_1^\dagger c_2 + c_2^\dagger c_1 \right] (\psi_1 |1_1, 0_2\rangle + \psi_2 |0_1, 1_2\rangle) = \\ = \gamma \left(\psi_1 |0_1, 1_2\rangle + \psi_2 |1_1, 0_2\rangle \right)$$

3. Project over $\langle \alpha_1, \alpha_2 |$ by using

$$\langle \alpha_1, \alpha_2 | \beta_1, \beta_2 \rangle = \delta_{\alpha_1, \beta_1} \delta_{\alpha_2, \beta_2}$$

   to obtain

$$\begin{cases} \epsilon_0 \psi_1 + \gamma \psi_2 = E \psi_1 \\ \\ \gamma \psi_1 + \epsilon_0 \psi_2 = E \psi_2 \end{cases}$$

4. Solve

$$\begin{pmatrix} \epsilon_0 & \gamma \\ \gamma & \epsilon_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

5. We obtain the known solutions

$$E_{\rm bond} = \epsilon_0 + \gamma, \qquad |\psi_{\rm bond}\rangle = \frac{1}{\sqrt{2}} \left[|1_1, 0_2\rangle + |0_1, 1_2\rangle \right]$$

$$E_{\rm anti} = \epsilon_0 - \gamma, \qquad |\psi_{\rm anti}\rangle = \frac{1}{\sqrt{2}} \left[|1_1, 0_2\rangle - |0_1, 1_2\rangle \right]$$

Note:

The action of the operators c_ν and $c_\mu^\dagger$ is defined by their $algebra$ given by the anti-commutation rules:

$$\{c_\mu^\dagger, c_\nu^\dagger\} = \{c_\mu, c_\nu\} = 0 \qquad \{c_\mu, c_\nu^\dagger\} = \delta_{\mu,\nu}$$

with

$$\{A, B\} = AB + BA$$

It then follows

1. The eigenvalues of $c_\nu^\dagger c_\nu$ are 0 or 1

$$\left(c_\nu^\dagger c_\nu\right)^2 = c_\nu^\dagger \left(c_\nu c_\nu^\dagger\right) c_\nu = c_\nu^\dagger \left(1 - c_\nu^\dagger c_\nu\right) c_\nu = c_\nu^\dagger c_\nu$$

$$\left.\begin{array}{c} c_\nu^\dagger c_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle \\ (c_\nu^\dagger c_\nu)^2 |n_\nu\rangle = n_\nu^2 |n_\nu\rangle \end{array}\right\} \rightarrow n_\nu = 0, 1$$

2. $c_\nu^\dagger c_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle \rightarrow c_\nu^\dagger c_\nu (c_\nu |n_\nu\rangle) = (1 - n_\nu) c_\nu |n_\nu\rangle$ with $n_\nu = 1$

3. $c_\nu^\dagger c_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle \rightarrow c_\nu^\dagger c_\nu (c_\nu^\dagger |n_\nu\rangle) = (1 - n_\nu) c_\nu^\dagger |n_\nu\rangle$ with $n_\nu = 0$

Second Quantization (very quick)

Let us go back to the time-dependent Schrödinger equation

$$i\hbar\dot{\Psi}(\mathbf{r}) = \left[-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{r})\right]\Psi(\mathbf{r})$$

This is an "equation of motion", which can be derived from the Lagrangian density

$$L = i\hbar\Psi^\dagger\dot{\Psi} - \frac{\hbar^2}{2m}\nabla\Psi^\dagger\cdot\nabla\Psi - U(\mathbf{r})\Psi^\dagger\Psi$$

If Ψ is the *dynamical variable* the conjugate momentum is

$$\pi = \frac{\partial L}{\partial\dot{\Psi}} = i\hbar\Psi^\dagger$$

The Hamiltonian density is given by

$$\mathcal{H} = \pi\dot{\Psi} - L = \frac{\hbar^2}{2m}\nabla\Psi^\dagger\cdot\nabla\Psi + U\Psi^\dagger\Psi$$

and the Hamiltonian is obtained by integrating

$$H = \int\mathcal{H}\,d^3\mathbf{r} = \int\Psi^\dagger\left(-\frac{\hbar^2\nabla^2}{2m} + U\right)\Psi\,d^3\mathbf{r}$$

Second quantization consists in making Ψ and $\Psi^\dagger$ operators (quantum fields) satisfying the following anti-commutation relations (for fermions)

$$\{\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$$
$$\{\Psi^\dagger(\mathbf{r}), \Psi^\dagger(\mathbf{r}')\} = \{\Psi(\mathbf{r}), \Psi(\mathbf{r}')\} = 0$$

This is due to Jordan and Wigner (1928). Particles satisfying the anticommutation rules are Fermions.

Now the operators $c_\nu^\dagger$ and c_ν can be introduced. Expand the wave function over a convenient basis. For a chain of H atom if $\phi_\nu(\mathbf{r})$ is the $1s$ orbital located at $\mathbf{r} - \mathbf{R}_\nu$

$$\Psi(\mathbf{r}) = \sum_\nu c_\nu \phi_\nu(\mathbf{r})$$

$$\Psi^\dagger(\mathbf{r}) = \sum_\nu c_\nu^\dagger \phi_\nu(\mathbf{r})$$

Finally from the anticommutation relations for the fields (integrating in $\mathbf{r}$):

$$\{c_\mu^\dagger, c_\nu^\dagger\} = \{c_\mu, c_\nu\} = 0 \qquad \{c_\mu, c_\nu^\dagger\} = \delta_{\mu,\nu}$$

What about the Hamiltonian?

$$H = \int \Psi^\dagger \left(-\frac{\hbar^2 \nabla^2}{2m} + U \right) \Psi \, d^3\mathbf{r}$$

$$H = \sum_{\mu,\nu} \left[\int \phi_\mu(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} + U \right) \phi_\nu(\mathbf{r}) \, d^3\mathbf{r} \right] c_\mu^\dagger c_\nu$$

$$\boxed{H = \sum_{\mu,\nu} H_{\mu\nu} c_\mu^\dagger c_\nu}$$

where $H_{\mu\nu}$ is the Hamiltonian matrix (as derived from the tight-binding method)

(For more details see: G.D. Mahan, *Many-Particle Physics*, Kluwer Academic)

Many-particle operators

The Coulomb electron-electron interaction depends on the position of two electrons

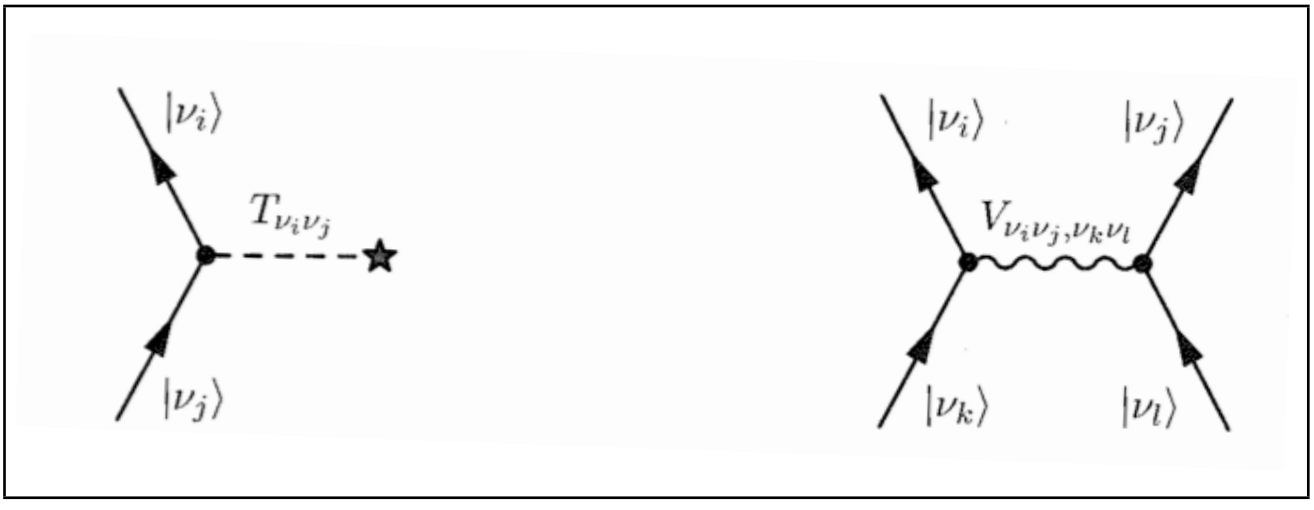
$$\frac{1}{2}\sum_i \sum_{j\neq i} V(\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{2}\sum_i \sum_{j\neq i} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

In second quantization this leads to the operator

$$H_{\mathrm{e-e}} = \frac{1}{2}\sum_{\nu_i\nu_j\nu_k\nu_l} V_{\nu_i\nu_j\nu_k\nu_l} c^\dagger_{\nu_i} c^\dagger_{\nu_k} c_{\nu_l} c_{\nu_j}$$

with

$$V_{\nu_i\nu_j\nu_k\nu_l} = \int d^3\mathbf{r}_i \int d^3\mathbf{r}_j \; \phi_{\nu_i}(\mathbf{r}_i)\phi_{\nu_j}(\mathbf{r}_i)\frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}\phi_{\nu_k}(\mathbf{r}_j)\phi_{\nu_l}(\mathbf{r}_j)$$



The Hubbard model for H_2

Consider the Hubbard Hamiltonian for H_2

$$H_{\rm U} = \epsilon_0 \sum_{\mu\sigma} \hat{n}_{\mu\sigma} + \gamma \sum_{\sigma} (c^{\dagger}_{1\sigma} c_{2\sigma} + c^{\dagger}_{2\sigma} c_{1\sigma}) + U \sum_{\mu} \hat{n}_{\mu\uparrow} \hat{n}_{\mu\downarrow}$$

where

1. $c^{\dagger}_{\mu\sigma}$

   creates an electron on the H atom at $\mathbf{R}_{\mu}$ with spin $\sigma = \uparrow, \downarrow$

2. $c_{\mu\sigma}$

   destroys an electron on the H atom at $\mathbf{R}_{\mu}$ with spin $\sigma = \uparrow, \downarrow$

3. $\epsilon_0 \sum_{\mu\sigma} \hat{n}_{\mu\sigma} + \gamma \sum_{\sigma} (c^{\dagger}_{1\sigma} c_{2\sigma} + c^{\dagger}_{2\sigma} c_{1\sigma})$

   is the standard (non-interacting tight-binding Hamiltonian for H_2)

4. $U \sum_{\mu} \hat{n}_{\mu\uparrow} \hat{n}_{\mu\downarrow}$

   is the on-site Coulomb repulsion

Consider the case of the H atom. Then

$$H_{\rm U} = \epsilon_0 \sum_\sigma \hat{n}_\sigma + U \hat{n}_\uparrow \hat{n}_\downarrow$$

		U=0	$U\neq 0$
No. e^-	$\|\Psi\rangle^-$		
0	$\|0\rangle^-$	0	0
1	$\|\uparrow\rangle^-$ $\|\downarrow\rangle^-$	ε_0	ε_0
2	$\|\uparrow\downarrow\rangle^-$	$2\varepsilon_0$	$2\varepsilon_0+U$

For interacting system the **spectrum depends on the number of electrons**

Solving the Hubbard model for H_2

Solve for different $N = n_\uparrow + n_\downarrow$

N=0

Only the trivial solution

$$E = 0, \qquad |\psi\rangle = 0$$

N=1

The wave-function can be written as

$$|\psi\rangle = \psi_1 |\uparrow, 0\rangle + \psi_2 |0, \uparrow\rangle$$

or

$$|\psi\rangle = \psi_1 |\downarrow, 0\rangle + \psi_2 |0, \downarrow\rangle$$

where

$$|\sigma, 0\rangle = c^\dagger_{1\sigma}|0, 0\rangle$$
$$|0, \sigma\rangle = c^\dagger_{2\sigma}|0, 0\rangle$$

this is completely equivalent to the case done at the very beginning (note that $U\sum_\mu \hat{n}_{\mu\uparrow}\hat{n}_{\mu\downarrow}|\sigma, 0\rangle = U\sum_\mu \hat{n}_{\mu\uparrow}\hat{n}_{\mu\downarrow}|0, \sigma\rangle = 0$)

The Hamiltonian matrix then is

$$\begin{pmatrix} \epsilon_0 & \gamma \\ \gamma & \epsilon_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

with solutions

$$E_{\rm bond} = \epsilon_0 + \gamma, \qquad |\psi_{\rm bond}\rangle = \frac{1}{\sqrt{2}} \left[|\sigma, 0\rangle + |0, \sigma\rangle \right]$$

$$E_{\rm anti} = \epsilon_0 - \gamma, \qquad |\psi_{\rm anti}\rangle = \frac{1}{\sqrt{2}} \left[|\sigma, 0\rangle - |0, \sigma\rangle \right]$$

and total energy

$$E_{\rm total} = \epsilon_0 + \gamma$$

Note that the solution has spin σ (1/2), but it is doubly degenerate.

N=2

There are six possible electronic configurations (states)

1. $|1\rangle = |\uparrow,\downarrow\rangle = c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}|0,0\rangle$

2. $|2\rangle = |\downarrow,\uparrow\rangle = c^\dagger_{1\downarrow}c^\dagger_{2\uparrow}|0,0\rangle$

3. $|3\rangle = |\downarrow\uparrow,0\rangle = c^\dagger_{1\downarrow}c^\dagger_{1\uparrow}|0,0\rangle$

4. $|4\rangle = |0,\downarrow\uparrow\rangle = c^\dagger_{2\downarrow}c^\dagger_{2\uparrow}|0,0\rangle$

5. $|5\rangle = |\uparrow,\uparrow\rangle = c^\dagger_{1\uparrow}c^\dagger_{2\uparrow}|0,0\rangle$

6. $|6\rangle = |\downarrow,\downarrow\rangle = c^\dagger_{1\downarrow}c^\dagger_{2\downarrow}|0,0\rangle$

Note that for $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, $n_\uparrow = n_\downarrow = 1$, i.e. they are spin singlet. Instead $|5\rangle$, $|6\rangle$ are spin triplet, i.e. $n_\sigma = 2$. Since there is no term in the Hamiltonian which flips the spin there are no matrix element between $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ and either $|5\rangle$ or $|6\rangle$. Now calculate the energy.

Triplets

$$|\psi\rangle = |5\rangle \quad \text{or} \quad |\psi\rangle = |6\rangle$$

and

$$E_{\rm total} = 2\epsilon_0$$

Singlets

The wave-function is then written on the $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ basis set

$$|\psi\rangle = \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle + \psi_4|4\rangle$$

again one writes

$$H_{\rm U}|\psi\rangle = E|\psi\rangle$$

and project over $|j\rangle$ $(j = 1, 2, 3, 4)$

This finally gives the matrix equation

$$\begin{pmatrix} 2\epsilon_0 & 0 & \gamma & \gamma \\ 0 & 2\epsilon_0 & \gamma & \gamma \\ \gamma & \gamma & 2\epsilon_0 + U & 0 \\ \gamma & \gamma & 0 & 2\epsilon_0 + U \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

The solutions are (eigenvalues and eigenvectors)

$$\varepsilon_1 = 2\epsilon_0 + \frac{U}{2} - \frac{\sqrt{U^2 + 16\gamma^2}}{2}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\{|1\rangle + |2\rangle\}\cos\theta + \{|3\rangle + |4\rangle\}\sin\theta\right]$$

$$\varepsilon_2 = 2\epsilon_0 + \frac{U}{2} + \frac{\sqrt{U^2 + 16\gamma^2}}{2}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[-\{|1\rangle + |2\rangle\}\sin\theta + \{|3\rangle + |4\rangle\}\cos\theta\right]$$

$$\varepsilon_3 = 2\epsilon_0$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

$$\varepsilon_4 = 2\epsilon_0 + U$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle),$$

where

$$\tan\theta = \frac{-4\gamma}{U + \sqrt{U^2 + 16\gamma^2}}.$$

The ground state energy is therefore the one given by $E_{\rm total} = \varepsilon_1$.

Note that:

1. For $\gamma = 0$ (two isolated H atoms)

$$\varepsilon_1 = 2\epsilon_0, \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$\varepsilon_2 = 2\epsilon_0 + U, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$$

$$\varepsilon_3 = 2\epsilon_0, \qquad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

$$\varepsilon_4 = 2\epsilon_0 + U, \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle),$$

2. For $U = 0$ (non-interacting)

$$\varepsilon_1 = 2(\epsilon_0 - \gamma), \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle - |3\rangle - |4\rangle)$$

$$\varepsilon_2 = 2(\epsilon_0 + \gamma), \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle + |3\rangle + |4\rangle)$$

$$\varepsilon_3 = 2\epsilon_0, \qquad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

$$\varepsilon_4 = 2\epsilon_0, \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle),$$

Note also that: One does not necessarily need to separate singlet and triplet. The full basis can be used:

$$|\psi\rangle = \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle + \psi_4|4\rangle + \psi_5|5\rangle + \psi_6|6\rangle$$

and the Hamiltonian matrix becomes

$$\begin{pmatrix} 2\epsilon_0 & 0 & \gamma & \gamma & 0 & 0 \\ 0 & 2\epsilon_0 & \gamma & \gamma & 0 & 0 \\ \gamma & \gamma & 2\epsilon_0 + U & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 2\epsilon_0 + U & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\epsilon_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\epsilon_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix}$$

N=3

The wave-function can be written as

$$|\psi\rangle = \psi_1|\uparrow, \uparrow\downarrow\rangle + \psi_2|\uparrow\downarrow, \uparrow\rangle$$

or

$$|\psi\rangle = \psi_1|\downarrow, \uparrow\downarrow\rangle + \psi_2|\uparrow\downarrow, \downarrow\rangle$$

where

$$|\sigma,\uparrow\downarrow\rangle = c^\dagger_{1\sigma}c^\dagger_{2\uparrow}c^\dagger_{2\downarrow}|0,0\rangle$$
$$|\uparrow\downarrow,\sigma\rangle = c^\dagger_{2\sigma}c^\dagger_{1\uparrow}c^\dagger_{1\downarrow}|0,0\rangle$$

this is similar to the $N=1$ case ($U\sum_\mu \hat{n}_{\mu\uparrow}\hat{n}_{\mu\downarrow}|\sigma,\uparrow\downarrow\rangle = U|\sigma,\uparrow\downarrow\rangle$)

The Hamiltonian matrix then is

$$\begin{pmatrix} 3\epsilon_0+U & \gamma \\ \gamma & 3\epsilon_0+U \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

with solutions

$$E_{\rm bond} = 3\epsilon_0+U+\gamma, \qquad |\psi_{\rm bond}\rangle = \frac{1}{\sqrt{2}}\left[|\sigma,\uparrow\downarrow\rangle + |\uparrow\downarrow,\sigma\rangle\right]$$

$$E_{\rm anti} = 3\epsilon_0+U-\gamma, \qquad |\psi_{\rm anti}\rangle = \frac{1}{\sqrt{2}}\left[|\sigma,\uparrow\downarrow\rangle - |\uparrow\downarrow,\sigma\rangle\right]$$

and total energy

$$E_{\rm total} = 3\epsilon_0+U+\gamma$$

Note that the solution has spin σ (1/2), but it is doubly degenerate.

N=4

Only the trivial solution

$$E_{\rm total} = 4\epsilon_0 + 2U, \qquad |\psi\rangle = |\uparrow\downarrow,\uparrow\downarrow\rangle$$

Let us summarize the spectrum found

N	spectrum	
0	0	
1	$\epsilon_0 + \gamma$	$\times 2$
	$\epsilon_0 - \gamma$	$\times 2$
2	$2\epsilon_0 + \frac{U}{2} - \frac{\sqrt{U^2+16\gamma^2}}{2}$	
	$2\epsilon_0 + \frac{U}{2} + \frac{\sqrt{U^2+16\gamma^2}}{2}$	
	$2\epsilon_0$	
	$2\epsilon_0 + U$	
	$2\epsilon_0$	
	$2\epsilon_0$	
3	$3\epsilon_0 + U + \gamma$	$\times 2$
	$3\epsilon_0 + U - \gamma$	$\times 2$
4	$4\epsilon_0 + 2U$	

How complicate can it get?

In general the wave-function need to be constructed over the basis defined by

$$|\Psi_\kappa\rangle = (\Pi_i^N \hat{c}_{i\sigma}^\dagger)|0\rangle, \tag{1}$$

The matrix elements of $H_{\rm U}$ are

$$H_{ij} = \langle\Psi_i|H_{\rm U}|\Psi_j\rangle. \tag{2}$$

For a L-sites and $n_\uparrow$ and $n_\downarrow$ electrons the number of basis function is

$$\kappa = \binom{L}{n_\uparrow}\cdot\binom{L}{n_\downarrow}, \quad \text{where,} \quad \binom{m}{n} = \frac{m!}{n!(m-n)!}. \tag{3}$$

The scaling is terrible!!

L	$n_\uparrow$	$n_\downarrow$	Hilbert Space dimension
2	1	1	4
4	2	2	36
6	3	3	400
8	4	4	4,900
10	5	5	63,504
12	6	6	853,776
14	7	7	11,778,624
16	8	8	165,636,900
18	9	9	2,363,904,400
20	10	10	34,134,779,536

System	L	N	Hilbert Space	Time	Memory
Na_2	2	2	4	1 μsec	128b
Na_6	6	6	400	1 sec	1.2Mb
Na_{10}	10	10	63,504	46 days	3.2Gb
Na_{12}	12	12	853,776	308 years	43Gb